

Polarization for arbitrary discrete memoryless channels

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Abstract—Channel polarization, originally proposed for binary-input channels, is generalized to arbitrary discrete memoryless channels. Specifically, it is shown that when the input alphabet size is a prime number, a similar construction to that for the binary case leads to polarization. This method can be extended to channels of composite input alphabet sizes by decomposing such channels into a set of channels with prime input alphabet sizes. It is also shown that all discrete memoryless channels can be polarized by randomized constructions. The introduction of randomness does not change the order of complexity of polar code construction, encoding, and decoding. A previous result on the error probability behavior of polar codes is also extended to the case of arbitrary discrete memoryless channels. The generalization of polarization to channels with arbitrary finite input alphabet sizes leads to polar-coding methods for approaching the true (as opposed to symmetric) channel capacity of arbitrary channels with discrete or continuous input alphabets.

Index Terms—Capacity-achieving codes, channel polarization, polar codes.

I. POLARIZATION

Channel polarization was introduced in [1] for binary input discrete memoryless channels as a coding technique to construct codes for data transmission. Polar codes are capable of achieving the ‘symmetric capacity’ of any binary input channel, using low-complexity encoding and decoding algorithms. In terms of the block-length N , polar codes can be encoded and decoded in complexity $O(N \log N)$ and achieve a block error probability that decays roughly like $2^{-\sqrt{N}}$. The latter result was shown in [2].

The aim of this note is to extend these results of [1], [2] to DMCs with q -ary inputs for any finite integer $q \geq 2$. To that end, we recall the polarization construction and outline how the results above were shown.

Given a binary input channel $W : \mathcal{X} \rightarrow \mathcal{Y}$ with $\mathcal{X} = \{0, 1\}$ define its *symmetric capacity* as

$$I(W) = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \frac{1}{2} W(y|x) \log_2 \frac{W(y|x)}{\sum_{x' \in \mathcal{X}} \frac{1}{2} W(y|x')}. \quad (1)$$

$I(W)$ is nothing but the mutual information developed between the input and the output of the channel when the input is uniformly distributed. In [1], two independent copies of W are first combined and then split so as to obtain two *unequal* binary input channels W^- and W^+ . The channel $W^2 : \mathcal{X}^2 \rightarrow \mathcal{Y}^2$ describes two uses of the channel W ,

$$W^2(y_1, y_2 | x_1, x_2) = W(y_1 | x_1) W(y_2 | x_2).$$

The input (x_1, x_2) to the channel W^2 is put in one-to-one correspondence with $(u_1, u_2) \in \mathcal{X}^2$ via $x_1 = (u_1 + u_2) \bmod 2$, $x_2 = u_2$, thus obtaining the *combined* channel $W_2 : \mathcal{X}^2 \rightarrow \mathcal{Y}^2$ described by

$$W_2(y_1, y_2 | u_1, u_2) = W(y_1 | u_1 + u_2) W(y_2 | u_2).$$

The *split* is inspired by the chain rule of mutual information: Let $U_1, U_2, X_1, X_2, Y_1, Y_2$ be random variables corresponding to their lowercase versions above. If U_1, U_2 are independent and uniformly distributed, then so are X_1, X_2 and consequently, on the one hand,

$$\begin{aligned} I(U_1, U_2; Y_1, Y_2) &= I(X_1, X_2; Y_1, Y_2) \\ &= I(X_1; Y_1) + I(X_2; Y_2) = 2I(W), \end{aligned}$$

and on the other

$$I(U_1, U_2; Y_1, Y_2) = I(U_1; Y_1, Y_2) + I(U_2; Y_1, Y_2, U_1).$$

The split channels W^- and W^+ describe those that occur on the right hand side of the equation above:

$$W^-(y_1, y_2 | u_1) = \sum_{u_2 \in \mathcal{X}} \frac{1}{2} W(y_1 | u_1 + u_2) W(y_2 | u_2),$$

$$W^+(y_1, y_2, u_1 | u_2) = \frac{1}{2} W(y_1 | u_1 + u_2) W(y_2 | u_2),$$

so that $I(U_1; Y_1, Y_2) = I(W^-)$ and $I(U_2; Y_1, Y_2, U_1) = I(W^+)$.

The polarization construction given in [1] is obtained by a repeated application of $W \mapsto (W^-, W^+)$. Since both W^- and W^+ are binary input channels, one can obtain $W^{--} := (W^-)^-$, $W^{-+} := (W^-)^+$, $W^{+-} := (W^+)^-$, and $W^{++} := (W^+)^+$. After n levels of application, one obtains 2^n channels $W^{--\dots-}, \dots, W^{++\dots+}$. The main observation in [1] is that these channels polarize in the following sense:

Proposition 1 ([1]). *For any $\delta > 0$,*

$$\lim_{n \rightarrow \infty} \frac{\#\{s \in \{+, -\}^n : I(W^s) \in (\delta, 1 - \delta)\}}{2^n} = 0. \quad (2)$$

In other words, except for a vanishing fraction, all the channels obtained at level n are either almost perfect, $I(W^s) \geq 1 - \delta$, or almost pure noise, $I(W^s) \leq \delta$.

As the equality $I(W^-) + I(W^+) = 2I(W)$ leads by induction to $\sum_{s \in \{+, -\}^n} I(W^s) = 2^n I(W)$, one then concludes that the fraction of almost perfect channels approaches the symmetric capacity. This last observation is the basis of what lets [1] conclude that polar codes achieve the symmetric capacity.

We give here a new proof of this proposition because it will readily generalize to the q -ary input case we will discuss later. Before we embark on this proof, we introduce the Bhattacharyya parameter for a binary input channel $W : \mathcal{X} \rightarrow \mathcal{Y}$, defined by

$$Z(W) = \sum_y \sqrt{W(y|0)W(y|1)}. \quad (3)$$

The relationship between $Z(W)$, $Z(W^-)$, $Z(W^+)$ and $I(W)$ is already discussed in [1], where the following is shown:

Lemma 1 ([1]).

- (i) $Z(W^+) = Z(W)^2$,
- (ii) $Z(W^-) \leq 2Z(W) - Z(W)^2$,
- (iii) $I(W) + Z(W) \geq 1$,
- (iv) $I(W)^2 + Z(W)^2 \leq 1$.

Proposition 1 was proved in [1] for the binary case ($q = 2$) using Lemma 1. Unfortunately, Lemma 1 does not generalize to the non-binary case ($q \geq 3$). The following alternate proof of Proposition 1 uses less stringent conditions that can be fulfilled for all $q \geq 2$.

Lemma 2. Suppose B_i , $i = 1, 2, \dots$ are i.i.d., $\{+, -\}$ -valued random variables with

$$P(B_1 = -) = P(B_1 = +) = \frac{1}{2}$$

defined on a probability space (Ω, \mathcal{F}, P) . Set $\mathcal{F}_0 = \{\phi, \Omega\}$ as the trivial σ -algebra and set \mathcal{F}_n , $n \geq 1$ to be the σ -field generated by (B_1, \dots, B_n) .

Suppose further that two stochastic processes $\{I_n : n \geq 0\}$ and $\{T_n : n \geq 0\}$ are defined on this probability space with the following properties:

- (i.1) I_n takes values in the interval $[0, 1]$ and is measurable with respect to \mathcal{F}_n . That is, I_0 is a constant, and I_n is a function of B_1, \dots, B_n .
- (i.2) $\{(I_n, \mathcal{F}_n) : n \geq 0\}$ is a martingale.
- (t.1) T_n takes values in the interval $[0, 1]$ and is measurable with respect to \mathcal{F}_n .
- (t.2) $T_{n+1} = T_n^2$ when $B_{n+1} = +$.
- (i&t.1) For any $\epsilon > 0$ there exists $\delta > 0$ such that $I_n \in (\epsilon, 1 - \epsilon)$ implies $T_n \in (\delta, 1 - \delta)$.

Then, $I_\infty := \lim_{n \rightarrow \infty} I_n$ exists with probability 1, I_∞ takes values in $\{0, 1\}$, and $P(I_\infty = 1) = I_0$.

Proof: See [4]. ■

Proof of Proposition 1: Let B_1, B_2, \dots be i.i.d., $\{+, -\}$ -valued random variables taking the two values with equal probability, as in Lemma 2. Define

$$I_n := I_n(B_1, \dots, B_n) = I(W^{B_1, \dots, B_n})$$

and

$$T_n := T_n(B_1, \dots, B_n) = Z(W^{B_1, \dots, B_n}).$$

It is easy to check that these processes satisfy the conditions of Lemma 2. Thus, the process I_n converges with probability 1 to a $\{0, 1\}$ -valued random variable. This implies that

$$\lim_{n \rightarrow \infty} P(I_n \in (\delta, 1 - \delta)) = 0.$$

Note that the distribution of (B_1, \dots, B_n) is the uniform distribution on $\{+, -\}^n$. Thus,

$$P(I_n \in (\delta, 1 - \delta)) = \frac{\#\{s \in \{+, -\}^n : I(W^s) \in (\delta, 1 - \delta)\}}{2^n},$$

and Proposition 1 follows. ■

The following lemma was proved in [2].

Lemma 3 ([2]). Suppose that the processes $\{B_n\}$, $\{I_n\}$ and $\{T_n\}$, in addition to the conditions (i.1), (i.2), (t.1), (t.2) and (i&t.1) in Lemma 2, also satisfy

- (t.3) For some constant κ , $T_{n+1} \leq \kappa T_n$ when $B_{n+1} = -$.
- (i&t.2) For any $\epsilon > 0$ there exists $\delta > 0$ such that $I_n > 1 - \delta$ implies $T_n < \epsilon$.

Then, for any $0 < \beta < 1/2$

$$\lim_{n \rightarrow \infty} P(T_n \leq 2^{-2^{\beta n}}) = I_0. \quad (4)$$

Note that in the proof of Proposition 1 the random variable T_n denotes the Bhattacharyya parameter of a randomly chosen channel after n steps of polarization. Therefore, Lemma 3 states that after n steps of polarization, almost all ‘good’ channels will have Bhattacharyya parameters that are smaller than $2^{-2^{\beta n}}$ for any $\beta < 1/2$, provided that n is sufficiently large. Since the Bhattacharyya parameter is an upper bound to the error probability of uncoded transmission, this implies that, at any fixed coding rate below $I_0 = I(W)$, the block error probability P_e of binary polar codes under successive cancellation decoding will satisfy

$$P_e \leq 2^{-N^\beta} \text{ for all } \beta < 1/2, \quad (5)$$

when the block-length $N = 2^n$ is sufficiently large.

II. POLARIZATION FOR q -ARY INPUT CHANNELS

In this section we will show how the transformation $(u_1, u_2) \mapsto (x_1, x_2)$ (and consequently $W \mapsto (W^-, W^+)$) and the definition of $Z(W)$ can be modified so that the hypotheses of Lemmas 2 and 3 are satisfied when the channel input alphabet is not binary. This will establish that the new transformation satisfies equation (2), leading to the conclusion that q -ary polar codes achieve symmetric capacity. That the error probability behaves roughly like $2^{-\sqrt{N}}$ will also follow.

To that end, let q denote the cardinality of the channel input alphabet \mathcal{X} and define

$$I(W) \triangleq \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} \frac{1}{q} W(y|x) \log \frac{W(y|x)}{\sum_{x' \in \mathcal{X}} \frac{1}{q} W(y|x')}$$

as the symmetric capacity of a channel W . We will take the base of the logarithm in this mutual information equal to q , so that $0 \leq I(W) \leq 1$.

For any pair of input letters $x, x' \in \mathcal{X}$, we define the Bhattacharyya distance between them as

$$Z(W_{\{x, x'\}}) = \sum_{y \in \mathcal{Y}} \sqrt{W(y|x)W(y|x')}. \quad (6)$$

Here, the notation $W_{\{x, x'\}}$ should be interpreted as denoting the channel obtained by restricting the input alphabet of W

to the subset $\{x, x'\} \subset \mathcal{X}$. We also define the average Bhattacharyya distance of W as

$$Z(W) = \sum_{x, x' \in \mathcal{X}, x \neq x'} \frac{1}{q(q-1)} Z(W_{\{x, x'\}}). \quad (7)$$

The average Bhattacharyya distance upper bounds the error probability of uncoded transmission:

Proposition 2. *Given a q -ary input channel W , let P_e denote the error probability of the maximum-likelihood decoder for a single channel use. Then,*

$$P_e \leq (q-1)Z(W).$$

Furthermore, I and Z are complementary, in the sense that one of them being close to 0 implies the other to be close to 1, and vice versa.

Proposition 3. *We have the following relationships between $I(W)$ and $Z(W)$.*

$$I(W) \geq \log \frac{q}{1 + (q-1)Z(W)} \quad (8)$$

$$I(W) \leq \log(q/2) + (\log 2)\sqrt{1 - Z(W)^2} \quad (9)$$

$$I(W) \leq 2(q-1)(\log e)\sqrt{1 - Z(W)^2}. \quad (10)$$

Proof of both these propositions can be found in [4].

A. Special case: Prime input alphabet sizes

We will see that when the input alphabet size q is a prime number, polarization can be achieved by similar constructions to the one for the binary case. For this purpose, we will equip the input alphabet \mathcal{X} with an operation $+$ so that $(\mathcal{X}, +)$ forms a group. (This is possible whether or not q is prime.) We will let 0 denote the identity element of $(\mathcal{X}, +)$. In particular, we may assume that $\mathcal{X} = \{0, \dots, q-1\}$ and that $+$ denotes modulo- q addition. Note that when q is prime, this is the only group of order q .

As in the binary case, we combine two independent copies of W , by choosing the input to each copy as

$$x_1 = u_1 + u_2, \quad x_2 = u_2. \quad (11)$$

We define the channels W^- and W^+ through

$$\begin{aligned} W^-(y_1, y_2 | u_1) &= \sum_{u_2 \in \mathcal{X}} \frac{1}{q} W_2(y_1, y_2 | u_1, u_2) \\ W^+(y_1, y_2, u_1 | u_2) &= \frac{1}{q} W_2(y_1, y_2 | u_1, u_2), \end{aligned} \quad (12)$$

where again $W_2(y_1, y_2 | u_1, u_2) = W(y_1 | u_1 + u_2)W(y_2 | u_2)$.

The main result of this section is the following:

Theorem 1. *The transformation described in (11) and (12) polarizes all q -ary input channels in the sense of Proposition 1, provided that q is a prime number. The rate of polarization under this transformation is the same as in the binary case, in the sense that the block error probabilities of polar codes based on this transformation satisfy (5).*

To prove Theorem 1 we first rewrite $Z(W)$ as

$$Z(W) = \frac{1}{q-1} \sum_{d \neq 0} Z_d(W),$$

where we define

$$Z_d(W) \triangleq \frac{1}{q} \sum_{x \in \mathcal{X}} Z(W_{\{x, x+d\}}), \quad d \neq 0.$$

We also define

$$Z_{\max}(W) \triangleq \max_{d \neq 0} Z_d(W).$$

Proof of Theorem 1: The proof is similar to the one for the binary case: Let B_1, B_2, \dots be i.i.d. $\{+, -\}$ -valued random variables taking the two values with equal probability. Define the random processes

$$I_n := I_n(B_1, \dots, B_n) = I(W^{B_1, \dots, B_n})$$

and

$$T_n := T_n(B_1, \dots, B_n) = Z_{\max}(W^{B_1, \dots, B_n}),$$

with $I_0 = I(W)$ and $T_0 = Z_{\max}(W)$. It is possible to show that $\{I_n\}$ and $\{T_n\}$ satisfy the conditions of Lemmas 2 and 3, the details of which may be found in [4]. Theorem 1 follows from this claim. ■

B. Arbitrary input alphabet sizes

The proof of Theorem 1 depends critically on the assumption that q is a prime number, and does not extend trivially to the case of composite input alphabet sizes. In fact, it is possible to find channels that the transformation given in the previous section will not polarize:

Example 1. *Consider the quaternary-input channel $W: \{0, 1, 2, 3\} \rightarrow \{0, 1\}$ defined by the transition probabilities $W(0 | 0) = W(0 | 2) = W(1 | 1) = W(1 | 3) = 1$, with $I(W) = \log 2$. If W is combined/split using the transformation described in (11) and (12), where $+$ denotes modulo-4 addition, then the channels W^+ and W^- are statistically equivalent to W . Therefore $I(W^-) = I(W) = I(W^+)$.*

For the general case, our first attempt at finding a polarizing transformation is to let

$$\begin{aligned} x_1 &= u_1 + u_2 \\ x_2 &= \pi(u_2) \end{aligned}$$

where $+$ denotes the group operation, and π is a fixed permutation on \mathcal{X} . In this case one can compute easily that

$$\begin{aligned} Z(W^+) &= \frac{1}{q(q-1)} \sum_{x, x': x \neq x'} Z(W_{\{\pi(x), \pi(x')\}}) \\ &\quad \frac{1}{q} \sum_u Z(W_{\{u+x, u+x'\}}). \end{aligned}$$

To be able to mimic the proof of Proposition 1 one would want that $Z(W^+) = Z(W)^2$. However, as the value of the inner sum above may depend on (x, x') , the equality $Z(W^+) = Z(W)^2$ will not necessarily hold in general.

As we will see, however, the average value of the above $Z(W^+)$ over all possible choices of π is $Z(W)^2$. For this reason, it is appropriate to think of a randomized channel combining/splitting operation, where the randomness is over the choice of π . To accomodate this randomness, again let (U_1, U_2) denote the independent and uniformly distributed inputs, and let Π be chosen uniformly at random from the set of permutations $\mathcal{P}_{\mathcal{X}}$, independently of (U_1, U_2) , and revealed to the receiver. Set

$$(X_1, X_2) = (U_1 + U_2, \Pi(U_2)). \quad (13)$$

Observe that

$$\begin{aligned} I(U_1, U_2; Y_1, Y_2, \Pi) &= 2I(W) \\ &= I(U_1; Y_1, Y_2, \Pi) + I(U_2; Y_1, Y_2, U_1, \Pi), \end{aligned}$$

and that we may define the channels $W^-: \mathcal{X} \rightarrow \mathcal{Y}^2 \times \mathcal{P}_{\mathcal{X}}$ and $W^+: \mathcal{X} \rightarrow \mathcal{Y}^2 \times \mathcal{X} \times \mathcal{P}_{\mathcal{X}}$ so that the terms on the right hand side equal $I(W^-)$ and $I(W^+)$:

$$W^-(y_1, y_2, \pi | u_1) = \sum_{u_2 \in \mathcal{X}} \frac{1}{q \cdot q!} W_2(y_1, y_2 | u_1, u_2) \quad (14)$$

$$W^+(y_1, y_2, u_1, \pi | u_2) = \frac{1}{q \cdot q!} W_2(y_1, y_2 | u_1, u_2), \quad (15)$$

where $W_2(y_1, y_2 | u_1, u_2) = W(y_1 | u_1 + u_2)W(y_2 | \pi(u_2))$.

Theorem 2. *The transformation described in (13), (14), and (15) polarizes all discrete memoryless channels W in the sense of Proposition 1.*

Proof: As in the binary case, we will let B_1, B_2, \dots be i.i.d., $\{+, -\}$ -valued random variables taking the two values with equal probability, and define

$$\begin{aligned} I_n &:= I_n(B_1, \dots, B_n) = I(W^{B_1, \dots, B_n}), \\ T_n &:= T_n(B_1, \dots, B_n) = Z(W^{B_1, \dots, B_n}), \end{aligned}$$

with $I_0 = I(W)$ and $T_0 = Z(W)$. One can check that the processes $\{I_n\}$ and $\{T_n\}$ satisfy the conditions of Lemma 2, and the theorem follows. ■

As $Z(W)$ upper bounds the error probability of uncoded transmission (cf. Proposition 2), in order to bound the error probability of q -ary polar codes in the form in (4) it suffices to show that the hypotheses of Lemma 3 hold. Since (i&t.2) is already implied by (9), it remains to show (t.3):

Proposition 4. *For the transformation described in (13), (14), and (15), we have*

$$\begin{aligned} Z(W) &\leq Z(W^-) \\ &\leq \min \{qZ(W), 2Z(W) + (q-1)Z(W)^2\}. \end{aligned}$$

We have seen that choosing the transformation $W \mapsto (W^-, W^+)$ in a random fashion from a set of transformations of size $q!$ yields $Z(W^+) = Z(W)^2$, leading to channel polarization. In particular, for each W there is at least one

transformation with $Z(W^+) \leq Z(W)^2$. Therefore, randomness is needed only in order to find such transformations at code construction stage, and not for encoding/decoding.

In a channel polarization construction of size N , there are $(2N-1)$ channels $(W, W^-, W^+, W^{--}, W^{-+}, \text{etc.})$ in the recursion tree of code construction. For each channel \tilde{W} residing in any one of the $(N-1)$ internal nodes of this tree, we need to find a suitable permutation π such that $Z(\tilde{W}^+) \leq Z(\tilde{W})^2$. Thus, the total complexity of finding the right permutations scales as $q!(N-1)$, in the worst case where all $q!$ permutations are considered. Recall that polar code construction also requires determining the frozen coordinates, which is a task of complexity $\Omega(N)$ at best. So, the order of polar code construction complexity is not altered by the introduction of randomization.

III. COMPLEMENTARY REMARKS

A. Reduction of randomness

The transformation $(u_1, u_2) \mapsto (x_1, x_2)$ described above uses a random permutation to satisfy $Z(W^+) = Z(W)^2$. This amount of randomness — over a set of size $q!$ — is in general not necessary, randomization over a set of size $(q-1)!$ is sufficient:

Theorem 3. *If the random permutation Π that defines (13) is chosen uniformly over the set of permutations for which 0 is a fixed point, the resulting transformation yields $Z(W^+) = Z(W)^2$ and thus is polarizing.*

A more significant reduction in randomness can be attained when the input alphabet \mathcal{X} can be equipped with operations $(+, \cdot)$ to form an algebraic field — this is possible if and only if q is a prime power. A random variable taking only $q-1$ values is sufficient in this case. (We have already seen that no randomization is needed when q is prime.) To see this, pick R to be uniformly distributed from the non-zero elements \mathcal{X}_* of \mathcal{X} , reveal it to the receiver and set

$$(x_1, x_2) = (u_1 + u_2, R \cdot u_2). \quad (16)$$

As was above we have

$$\begin{aligned} 2I(W) &= I(U_1, U_2; Y_1, Y_2, R) \\ &= I(U_1; Y_1, Y_2, R) + I(U_2; Y_1, Y_2, U_1, R) \\ &= I(W^-) + I(W^+) \end{aligned}$$

provided that we define $W^-: \mathcal{X} \rightarrow \mathcal{Y}^2 \times \mathcal{X}_*$ and $W^+: \mathcal{X} \rightarrow \mathcal{Y}^2 \times \mathcal{X} \times \mathcal{X}_*$ as

$$\begin{aligned} W^-(y_1, y_2, r | u_1) &= \frac{1}{q(q-1)} \sum_{u_2 \in \mathcal{X}} W(y_1 | u_1 + u_2) W(y_2 | r \cdot u_2), \quad (17) \\ W^+(y_1, y_2, u_1, r | u_2) &= \frac{1}{q(q-1)} W(y_1 | u_1 + u_2) W(y_2 | r \cdot u_2). \end{aligned} \quad (18)$$

Theorem 4. *The transformation described in (16), (17), and (18) polarizes all q -ary input channels in the sense of Proposition 1, provided that q is a prime power.*

Proof: One can check that $Z(W^+) = Z(W)^2$, from which the theorem follows via Lemma 2 along the path as Proposition 1. ■

It is also easy to check that the hypotheses of Lemma 3 remains valid with, and thus the bound on the error probability (4) holds for this particular polarization scheme too.

When the field is of odd characteristic (i.e., when q is not a power of two), a further reduction is possible: since $\sum_{u'} Z(W_{\{u', u'+d\}})$ is invariant under $d \rightarrow -d$, one can show that the range of R can be reduced from \mathcal{X}_* to only half of the elements in \mathcal{X}_* , by partitioning \mathcal{X}_* into two equal parts in one-to-one correspondence via $r \mapsto -r$, and picking one of the parts as the range of R . It is easy to show that choosing R uniformly at random over this set of size $(q-1)/2$ will also yield $Z(W^+) = Z(W)^2$.

B. A method to avoid randomness

When the input alphabet size q is not prime, an alternative multi-level code construction technique can be used in order to avoid randomness: Consider a channel W with input alphabet size $q = \prod_{i=1}^L q_i$, where q_i 's are the prime factors of q . When the input X to W is uniformly distributed on \mathcal{X} , one can write $X = (U_1, \dots, U_L)$, where U_i 's are independent and uniformly distributed on their respective ranges $\mathcal{U}_i = \{0, \dots, q_i - 1\}$. Defining the channels $W^{(i)} : \mathcal{U}_i \rightarrow \mathcal{Y} \times \mathcal{U}_1 \times \dots \times \mathcal{U}_{i-1}$ through

$$W^{(i)}(y, u_1^{i-1} | u_i) = \prod_{j \neq i} q_j^{-1} \sum_{u_{i+1}^L} W(y | (u_1^L)),$$

it is easily seen that

$$\begin{aligned} I(W) &= I(X; Y) = I(U_1^L; Y) \\ &= \sum_i I(U_i; Y, U_1^{i-1}) = \sum_i I(W^{(i)}). \end{aligned}$$

Having decomposed W into $W^{(1)}, \dots, W^{(L)}$, one can polarize each channel $W^{(i)}$ separately. The order of successive cancellation decoding in this multi-level construction is to first decode all channels derived from $W^{(1)}$, then all channels derived from $W^{(2)}$, and so on. Since the input alphabet size of each channel is prime, no randomization is needed.

C. Equidistant channels

A channel W is said to be *equidistant* if $Z(W_{\{x, x'\}})$ is constant for all pairs of distinct input letters x and x' . These are channels with a high degree of symmetry. In particular, if a channel W is equidistant, then so are the channels W^+ and W^- created by the deterministic mapping $(u_1, u_2) \mapsto (u_1 + u_2, u_2)$. By similar arguments to those in Section II-A, it follows that this mapping polarizes equidistant channels, regardless of the input alphabet size.

D. How to achieve channel capacity using polar codes

In all of the above, the input letters of the channel under consideration were used with equal frequency. This was sufficient to achieve the symmetric channel capacity. However, in order to achieve the true channel capacity, one should be able to use the channel inputs with non-uniform frequencies in general. The following method, discussed in [3, p. 208], shows how to implement non-uniform input distributions within the polar coding framework.

Given two finite sets \mathcal{X} and \mathcal{X}' with $m = |\mathcal{X}'|$, any distribution P_X on \mathcal{X} for which $mP_X(x)$ is an integer for all x can be induced by the uniform distribution on \mathcal{X}' and a deterministic map $f : \mathcal{X}' \rightarrow \mathcal{X}$.

Given a channel $W : \mathcal{X} \rightarrow \mathcal{Y}$, and a distribution P_X as above, we can construct the channel $W' : \mathcal{X}' \rightarrow \mathcal{Y}$ whose input alphabet is \mathcal{X}' and $W'(y|x') = W(y|f(x'))$. Then, $I(W')$ is the same as the mutual information developed between the input and output of the channel W when the input distribution is P_X . Consequently, a method that achieves the symmetric capacity of any discrete memoryless channel, such as the channel polarization method considered in this paper, can be extended to approach the true capacity of any discrete memoryless channel by taking P_X as a rational distribution approximating the capacity achieving distribution. (In order to avoid randomization, one may use prime m in the constructions.)

E. Channels with continuous alphabets

Although the discussion above has been restricted to channels with *discrete* input and output alphabets, it should be clear that the results hold when the output alphabet is continuous, with minor notational changes. In the more interesting case of channels with continuous input alphabets — possibly with input constraints, such as the additive Gaussian noise channel with an input power constraint — we may readily apply the method of Section III-D to approximate any desired continuous input distribution for the target channel, and thereby approach its capacity using polar codes.

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